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The diagonal cohomology of the universal Steenrod algebra

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Abstract

In this paper we compute the diagonal cohomology of the algebra Q of cohomology operations in the category of H_∞-ring spectra, also known as the universal Steenrod algebra. Our methods involve results about Koszul algebras. It turns out that $D^*(Q)$ is isomorphic to a suitable completion of Q itself. © 1997 Published by Elsevier Science B.V.

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1. Introduction and statement of the results

We recall that the mod 2 universal Steenrod algebra Q can be presented as follows:

$$Q = \left\langle x_k \mid x_{2k-1-n}x_k = \sum_j \binom{n-1-j}{j} x_{2k-1-j}x_{k+j-n} \right\rangle, \tag{1}$$

with $k \in \mathbb{Z}$ and $n \in \mathbb{N}_0$. The algebra Q is an interesting object of investigation, as it contains Λ, the lambda algebra introduced in [1], as a subalgebra. Moreover, the Steenrod algebra is a quotient of Q. It would be nice to compute the cohomology algebra $H(Q) =: \mathrm{Ext}_Q(\mathbb{F}_2, \mathbb{F}_2)$, but this problem is presently unsolved. In this paper we only succeed in giving a description of the so-called *diagonal* cohomology of Q. As $H(Q)$ is a naturally bigraded object, we set

$$D^k(Q) =: H^{kk}(Q)$$

and find that

$$D^*(Q) \cong Q^\wedge$$

the completion of Q itself with respect to a certain chain of ideals.

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2. Cohomology of algebras

Let T be the tensor algebra (on a field $\mathbb{F}$) over the set X. In the sequel X will always be of the form

$$X = \{x_i \mid i \in \mathscr{I}\},$$

where $\mathscr{I} \subseteq \mathbb{Z}$. Moreover, we assume $\mathbb{F} = \mathbb{F}_2$, although everything works in a more general contest. Let T_j be the $\mathbb{F}$-vector space spanned by the words (or monomials) on X of length j, for each $j \in \mathbb{N}$. We set $T_0 = \mathbb{F}$. An augmentation

$$\varepsilon : T \longrightarrow \mathbb{F}$$

is obtained by setting $\varepsilon_{|T_0} = id_{\mathbb{F}}$ and $\varepsilon(\alpha) = 0$ for each monomial α of positive length. Clearly the unit

$$\eta : \mathbb{F} \longrightarrow T$$

is just the inclusion into T_0. If A is an augmented algebra with unit over $\mathbb{F}$, a presentation of A is an augmented epimorphism

$$\pi : T \longrightarrow A,$$

where T is the tensor algebra over some suitable set X. If we set $y_i = \pi(x_i)$, we also say that A is presented by generators $\{y_i \mid i \in \mathscr{I}\}$ and relations $f(y_{i_1}, \ldots, y_{i_k}) = 0$, where $f(x_{i_1}, \ldots, x_{i_k}) \in \ker \pi$.

Example. Let $\mathscr{A}$ be the Steenrod algebra and $\mathscr{I} = \mathbb{N}_0$. Then

$$\pi : x_i \in T \longmapsto Sq^i \in \mathscr{A}$$

is a presentation of $\mathscr{A}$ and the elements of $\ker \pi$ are polynomial expressions in the x_i's which are combinations of expressions of the form

$$x_0 + 1 \quad \text{or} \quad x_a x_b + \sum_j \binom{b-1-j}{a-2j} x_{a+b-j} x_j, \quad a < 2b\ .$$

We will say that A is homogeneous if the polynomial expressions $f(x_{i_1}, \ldots, x_{i_k}) \in \ker \pi$ are homogeneous. For instance, $\mathscr{A}$ is not homogeneous, because of the relation $x_0 + 1$. Let $J = \ker \varepsilon$ be the augmentation ideal.

Definition. Let

$$\bar{B}(A) =: T(J) = \bigoplus_{k \in \mathbb{N}_0} \underbrace{J \otimes \ldots \otimes J}_{k}\ .$$

We define a map

$$\partial : \bar{B}_s(A) \longrightarrow \bar{B}_{s-1}(A)$$

by setting

$$\partial(a_1 \otimes \ldots \otimes a_s) = \sum_{i=2}^{s} a_1 \otimes \ldots \otimes a_{i-1}a_i \otimes \ldots \otimes a_s.$$

$(\bar{B}(A), \partial)$ is a chain complex, known as the reduced *bar* construction, and from homological algebra we know that it computes the homology and the cohomology of A, i.e.

$$H_*(A) =: \mathrm{Tor}_A(\mathbb{F}, \mathbb{F})$$

is the homology of the chain complex $(\bar{B}(A), \partial)$ and

$$H^*(A) =: \mathrm{Ext}_A(\mathbb{F}, \mathbb{F})$$

is the homology of the dual cochain complex $(\bar{C}(A), \delta)$, which is called the reduced *cobar* complex. If $f \in \bar{C}^s(A)$ is a cochain, we have

$$(\delta f)(m) =: f(\partial(m)) \quad \forall\ m \in \bar{B}_{s+1}(A).$$

Let

$$\mu^* : A^* \longrightarrow A^* \otimes A^*$$

be the comultiplication in A^*, dual to the multiplication $\mu : A \otimes A \to A$. If

$$\alpha_1 \otimes \ldots \otimes \alpha_n \in \bar{C}^n(A),$$

where each α_i is dual to some element $a_i \in J$, and we write

$$\mu^*(\alpha_i) = \sum_r \alpha'_{i,r} \otimes \alpha''_{i,r},$$

we have

$$\delta(\alpha_1 \otimes \ldots \otimes \alpha_n) = \sum_i \left(\alpha_1 \otimes \ldots \otimes \alpha_{i-1} \otimes \sum_r (\alpha'_{i,r} \otimes \alpha''_{i,r}) \otimes \alpha_{i+1} \otimes \ldots \otimes \alpha_n\right).$$

In particular

$$\delta(\alpha) = \mu^*(\alpha), \quad \alpha \in J^* = \bar{C}^1(A).$$

In $\bar{C}(A)$ a graded product (*cup product*) is defined in the usual way. In general we will write $\alpha(i_1, \ldots, i_k)$ for the element of A^* dual to the monomial $x_{i_1} \ldots x_{i_k}$. Assume A is homogeneous and let $\mathscr{B}$ be a linear basis of monomials of A. If $x_{i_1} \ldots x_{i_k} \in \mathscr{B}$, the string $I = (i_1, \ldots, i_k)$ will be called the *label* of $x_{i_1} \ldots x_{i_k}$, and we write $x_I, \alpha(I)$ instead of $x_{i_1} \ldots x_{i_k}, \alpha(i_1, \ldots, i_k)$. Let S be the set of all the labels of the monomials of $\mathscr{B}$. As A is homogeneous and $\mathscr{B}$ is a linear basis of monomials, any monomial $x_a x_b$ of length 2 can uniquely be expressed as

$$x_a x_b = \sum_{(c,d) \in S} f(a,b;c,d) x_c x_d, \quad f \in \mathbb{F}. \tag{2}$$

The above formula (2) is called the admissible expression of $x_a x_b$. We have that

$$\delta(\alpha(i,j)) = \begin{cases} \alpha(i)\alpha(j) + \sum_{(h,k)\notin S} f(h,k;i,j)\alpha(h)\alpha(k) & \text{if } (i,j) \in S, \\ \alpha(i)\alpha(j) & \text{if } (i,j) \notin S. \end{cases} \tag{3}$$

As A is graded by length, $H^*(A)$ is bigraded: the first grading is given by the homological degree, the second by the length. We set

$$D^k(A) =: H^{kk}(A).$$

The direct sum

$$D^*(A) = \bigoplus_{k\geq 0} D^k(A)$$

is stable with respect to the cup product and is therefore a subalgebra of $H^*(A)$, the *diagonal* cohomology algebra of A.

Remark. There is an important class of algebras, the pre-Koszul algebras, characterised by the fact that they admit a presentation with a minimal set of relations only involving monomials of length less than or equal to 2. A homogeneous pre-Koszul algebra is actually a Koszul algebra if $D^*(A) = H^{**}(A)$. The theory of Koszul algebras is closely related to the results presented in the sequel (see [6]).

3. The universal Steenrod algebra

By (mod 2) universal Steenrod algebra we mean the algebra Q of cohomology operations in the category $\mathscr{C}(2,\infty)$ of H_∞-ring spectra (see [5]). In [3, 4] we gave an invariant theoretical description of Q and also a presentation (1) in terms of generators and relations which generalise the Adem relations of the mod 2 Steenrod algebra $\mathscr{A}$ and the defining relations of the Λ-algebra of [1]. As we have already seen, Q can be presented by a set of generators $\{x_j \mid j \in \mathbb{Z}\}$ and relations

$$x_{2k-1-n}x_k = \sum_j \binom{n-1-j}{j} x_{2k-1-j}x_{k+j-n}, \quad k \in \mathbb{Z},\ n \in \mathbb{N}_0, \tag{4}$$

which we call the *generalised Adem relations*. As we point out in [4], there are two important families of algebras which are closely related to Q. For each $r \in \mathbb{Z}$ we let L_r be the subalgebra of Q generated by $x_r, x_{r-1}, x_{r-2}, \ldots$ and set $K^r =: Q/I(r)$, where $I(r)$ is the two-sided ideal of Q generated by $x_{r-1}, x_{r-2}, x_{r-3}, \ldots$. We point out that

(i) $L_0 \cong \Lambda$;
(ii) $L_1 \cong \bar{\Lambda}$ (the algebra introduced in [2]);
(iii) $K^0/I \cong \mathscr{A}$ (where $I = (x_0 + 1)$);
(iv) $K^1 \cong \mathscr{A}_L$ (the Steenrod algebra for the category of simplicial Lie algebras).

This is a motivation for the study of the cohomology of Q and its relations with the Adams spectral sequence.

4. Cohomology computations

Let us compute the diagonal cohomology of L_r. As L_r is a subalgebra of Q, L_r can be presented by generators $x_r, x_{r-1}, x_{r-2}, \dots$ and relations of two types. Firstly, we have all the generalised Adem relations which only involve such generators, i.e.

$$x_{2k-1-n}x_k = \sum_j \binom{n-1-j}{j} x_{2k-1-j}x_{k+j-n}, \quad k,\ 2k-1 \le r.$$

Moreover, there are relations which involve monomials of length greater than 2, which are still homogeneous and are combinations of generalised Adem relations involving generators with an index higher than r. For example, we have

$$x_2x_2 = x_3x_1; \quad x_1x_1 = 0 \qquad \text{in } Q.$$

Hence

$$x_2x_2x_1 = x_3x_1x_1 = 0 \qquad \text{in } Q.$$

Therefore

$$x_2x_2x_1 = 0 \qquad \text{in } L_2 \tag{5}$$

as L_2 is a subalgebra of Q. But the relation $x_2x_2 = x_3x_1$ cannot appear in a presentation of L_2 as $x_3 \notin L_2$. As a consequence, we need to impose the relation (5) which involves monomials of length 3. A basis of monomials for L_r is given by the admissible monomials and some other monomials which are not admissible but cannot be expressed as sums of admissibles (because some Adem relations are missing in L_r). For example, a nonadmissible monomial $x_{2k-1-n}x_k$ (with $k, 2k-1-n \le r$) is basic if and only if $2k-1 > r$. In fact, if $k,\ 2k-1 \le r$ then the relation (4) appears in the presentation of L_r and $x_{2k-1-n}x_k$ is not basic (it is expressable as a sum of admissibles). On the other hand, if $2k-1 > r$ but $k,\ 2k-1-n \le r$, we have that $x_{2k-1-n}x_k \in L_r$ but (4) does not appear in the presentation of L_r, i.e. $x_{2k-1-n}x_k$ cannot be expressed as a sum of admissible monomials. Hence such monomial is either basic or else it admits an expression as a sum of monomials (but not all of them are admissible). This latter situation does not actually occur, as each nonadmissible monomial has a different top term in its admissible expression, as in (4) the summand $\binom{n-1}{0}x_{2k-1}x_{k-n}$, which corresponds to $j = 0$, never vanishes, being $\binom{n-1}{0} = 1$.

In order to compute $D^*(L_r)$, we notice that

$$D^k(L_r) = H^{k,k}(L_r) = \frac{\ker[\delta^{k,k} : \bar{C}^{k,k}(L_r) \to \bar{C}^{k+1,k}(L_r)]}{\operatorname{im}[\delta^{k-1,k} : \bar{C}^{k-1,k}(L_r) \to \bar{C}^{k,k}(L_r)]}$$
$$= \frac{C^{k,k}(L_r)}{\operatorname{im} \delta^{k-1,k}}\ .$$

$C^{k-1,k}(L_r)$ is spanned by cochains of the form

$$\alpha = \alpha(i_1) \otimes \cdots \otimes \alpha(i_j, i_{j+1}) \otimes \cdots \otimes \alpha(i_k).$$

Hence $D^*(L_r)$ is presented by generators $\alpha(i)$, $i \leq r$ and relations of the form $\delta^{1,2}(\alpha(i,j)) = 0$. If $x_i x_j$ is not admissible, but it is basic in L_r (i.e. $i, j \leq r$ but $2j - 1 > r$), $x_i x_j$ does not appear in the expression of any other monomial of length 2 as a sum of basic elements. Hence, from (2) and (3), we have that

$$\delta\big(\alpha(i,j)\big) = \alpha(i) \otimes \alpha(j).$$

Assume now $i \geq 2j$ and write $i = 2j + \ell$, $\ell \geq 0$. From (3), we see that

$$\delta(\alpha(2j+\ell,j)) = \alpha(2j+\ell) \otimes \alpha(j) + \sum_h f(j,\ell,h)\alpha(2j+\ell-h) \otimes \alpha(j+h) \tag{6}$$

with h such that $j+h \leq r$, where $\alpha(2j+\ell-h) \otimes \alpha(j+h)$ corresponds to a nonadmissible monomial $x_{2j+\ell-h}x_{j+h}$, i.e. h is such that $2j+\ell-h < 2j+2h$, and $f(j,\ell,h)$ is the coefficient of $x_{2j+\ell}x_j$ in the admissible expression of $x_{2j+\ell-h}x_{j+h}$. As $x_{2j+\ell-h}x_{j+h}$ is not admissible, there exists $n \in \mathbb{N}_0$ such that

$$2j + \ell - h = 2j + 2h - 1 - n.$$

In other words $n = 3h - \ell - 1$. Let us look at the generalised Adem relation

$$x_{2j+\ell-h}x_{j+h} = x_{2(j+h)-1-n}x_{j+h} = \sum \binom{3h-\ell-2-p}{p} x_{2(j+h)-1-p}x_{j+2p-n}. \tag{7}$$

We find the monomial $x_{2j+\ell}x_j$ in the sum on the RHS of (7) when $2(j+h)-1-p = 2j+\ell$ i.e. $p = 2h-\ell-1$. Hence its coefficient is

$$\binom{3h-\ell-2-2h+\ell+1}{2h-\ell-1} = \binom{h-1}{2h-\ell-1}.$$

The generating relations (6) of $D^*(L_r)$ are therefore

$$\alpha(2j+\ell) \otimes \alpha(j) = \sum_h \binom{h-1}{2h-\ell-1} \alpha(2j+\ell-h) \otimes \alpha(j+h) \tag{8}$$

where

$$\ell \geq 0\,; \quad j+\ell, 2j+\ell \leq r.$$

The sum runs over those indices h such that $j+h \leq r$ and we set

$$\alpha(2j+\ell-h) \otimes \alpha(j+h) = 0$$

in the RHS of (8) if $2j+\ell-h > r$ or $j+h > r$. We summarize the above computation in the following statement:

Proposition. *For each integer r, the diagonal cohomology algebra $D^*(L_r)$ of the subalgebra L_r of Q is presented by the set of generators $\{\alpha(i) \mid i \leq r\}$ and relations* (8) *and $\alpha(i) \otimes \alpha(j) = 0$ if $x_i x_j$ is basic but not admissible in L_r.*

We now look at $D^*(Q)$. For each integer r, let

$$i_r : L_r \longrightarrow L_{r+1}$$

be the inclusion. We have

$$Q = \operatorname*{colim}_{r\to\infty} \{ L_r, i_r \}.$$

Proposition.

$$H^*(Q) \cong \lim_{\infty\leftarrow r} \{ H^*(L_r), i_r^* \}.$$

Proof. We have a short exact sequence in cohomology

$$\begin{array}{l} 0 \longrightarrow \lim\limits_{\infty\leftarrow r}{}^1 \{H^*(L_r), i_r^*\} \\ \quad \longrightarrow H^*\big(\operatorname*{colim}\limits_{r\to\infty}\{L_r, i_r\}\big) \xrightarrow{\Phi} \lim\limits_{\infty\leftarrow r} \{H^*(L_r), i_r^*\} \longrightarrow 0 \\ \qquad\qquad \Big\| \\ \qquad\quad H^*(Q) \end{array} \tag{9}$$

As L_r is finite in each bidegree and i_r preserves all the gradings, the Mittag–Leffler conditions are satisfied and the $\lim^1$ term vanishes in (9). Therefore Φ is an isomorphism. □

Corollary 1.

$$D^*(Q) \cong \lim_{\infty\leftarrow r} \{ D^*(L_r), i_r^* \}.$$

Proof. Obvious, by restriction. □

We now observe that for each integer s the quotient algebra

$$K^s = Q/(x_{s-1}, x_{s-2}, x_{s-3}, \ldots)$$

can be presented by generators x_s, x_{s+1}, x_{s+2}, ... and relations (4), with the convention that a monomial $x_i x_j$ has to be considered zero if i or j is less than s.

Lemma 1. *For each $r \in \mathbb{Z}$ there is an epimorphism*

$$\omega_r : K^{-r+1} \longrightarrow D^*(L_r).$$

Proof. Define

$$\omega_r : K^{-r+1} \longrightarrow D^*(L_r)$$

by setting $\omega_r(x_j) = \big(\alpha(-j+1)\big)$. It is straightforward to check that ω_r takes relations of K^{-r+1} to relations of the form (8) of $D^*(L_r)$. The fact that ω_r is epic is trivial. □

We point out that we do not know whether ω_r is an isomorphism. If so, the proof of the next theorem would become easier.

Let $\pi_s : K^s \twoheadrightarrow K^{s+1}$ be the epimorphism defined by setting

$$\pi_s(x_j) = \begin{cases} x_j & \text{for } j > s, \\ 0 & \text{for } j = s. \end{cases}$$

Lemma 2. *For each integer r, the following diagram commutes:*

$$\begin{array}{ccc} D^*(L_r) & \xleftarrow{\;i_r^*\;} & D^*(L_{r+1}) \\ {\scriptstyle \omega_r}\uparrow & & \uparrow{\scriptstyle \omega_{r+1}} \\ K^{-r+1} & \xleftarrow[\pi_{-r}]{} & K^{-r} \end{array}$$

Proof. Again, it is an easy verification on the generators. □

We can now consider the inverse system $\{K^s, \pi_s\}$ and the increasing chain

$$\dots, I(s), I(s+1), I(s+2), \dots \tag{10}$$

of ideals of Q. We have that

$$\lim_{-\infty \leftarrow s} \{K^s, \pi_s\} \cong Q\hat{}\,,$$

where $Q\hat{}$ is the completion of Q with respect to the chain (10) of ideals. We conclude with the following announced result.

Theorem.

$$D^*(Q) \cong Q\hat{}\,.$$

Proof. From Corollary 1, Lemmas 1 and 2, we have that

$$D^*(Q) \cong \lim_{\infty \leftarrow r} \{D^*(L_r), i_r^*\}$$

and the sequence $\{\omega_r\}$ defines a homomorphism

$$\omega : Q\hat{} \longrightarrow D^*(Q)$$

of inverse limits, and it is easy to see that ω is epic. We want to check that ω is monic as well. Let $\beta = \{\beta_h\}_{h \in \mathbb{Z}} \in Q\hat{}$ be an admissible sequence, i.e.

$$\beta_h \in K^{-h+1} \qquad \text{and} \qquad \beta_h = \pi_{-h}(\beta_{-h+1}).$$

Suppose that $\beta_r \neq 0$ and $\omega_r(\beta_r) = 0 \in D^*(L_r)$. As

$$\ker \omega_r = (x_{-2k+2+n} x_{-k+1} \mid k, 2k-1-n \leq r,\ 2k-1 > r)$$

β_r is a sum of monomials of the form $x_{-2k+2+n} x_{-k+1}$ with $k, 2k-1-n \leq r$, $2k-1 > r$. For m large enough, the relation (4) appears in the presentation of L_{r+m} and $\omega_{r+m}(\beta_{r+m}) \neq 0$. We deduce that from $\beta \neq 0$ it follows that $\omega(\beta) \neq 0$ and ω is monic. □

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